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Persistence and Stability for a Three-Species Ratio-Dependent Predator-Prey System

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ABSTRACT

In this paper we study some qualitative properties such as persistence and stability for a three-species ratio-dependent predator-prey system with time delay in a three-patch environment. It is shown that the system is permanent under some suitable conditions.

Keywords: Predator-Prey Model; Time Delay; Diffusion; Uniform Persistence.

1. INTRODUCTION

Although the predator-prey theory has seen much progress in the last five decades, many long standing mathematical and ecological problems remain open (Rui and LanSun, 2000).

Since the pioneering theoretical work by (Skellam, 1951), many papers have focused on the effect of spatial factors, which plays a crucial role in permanence and stability, of population (Leung, 1987; Rothe, 1976). In fact, the dispersal between patches often occurs in ecological environments, and more realistic model should include the dispersal process. Many authors have studied the permanence and stability of Lotka-Volterra diffusion models (El-Owaidy and Ismail, 2003; Freedman and Takeuchi, 1989; Lu and Takeuchi, 1992). In addition, it is generally recognized that some kind of time delays are inevitable in population interactions and tend to be destabilizing in the sense that longer delays may destroy the stability of positive equilibrium (see (Cushing, 1977; Freedman and Tackeuchi, 1989) and the reference cited therein).

Time delay due to gestation is among them, because generally duration of τ time units elapses when an individual prey is killed and the moment when the corresponding increase in the predator population is realized. The effect of this kind of delay on the asymptotic behavior of populations has been studied by a number of papers (see, for example (Wang and Ma, 1997).

In this paper, we incorporate time delay due to gestation into the ratio-dependent predator-prey diffusion system. For the three-species ratio-dependent predator-prey model with diffusion and Michaelis-Menten type functional response, this results in the following delayed system:

$$\left.\begin{aligned}
\dot{x}_1 &= x_1(t)\left(a_1 - a_{11}x_1(t) - \frac{a_{13}x_3(t)}{mx_3(t)+x_1(t)}\right) \\
&\qquad + D_1\big(x_2(t)+x_4(t)-x_1(t)\big),\\
\dot{x}_2 &= x_2(t)\big(a_2 - a_{22}x_2(t)\big) + D_2\big(x_1(t)+x_4(t)-x_2(t)\big),\\
\dot{x}_3 &= x_3(t)\left(-a_3 + \frac{a_{31}x_1(t-\tau)}{mx_3(t-\tau)+x_1(t-\tau)}\right),\\
\dot{x}_4 &= x_4(t)\big(a_4 - a_{44}x_4(t)\big) + D_4\big(x_1(t)+x_2(t)-x_4(t)\big),
\end{aligned}\right\} \quad (1.1)$$

where $x_i(t)$ represents the prey population in the i^{th} patch, $i = 1,2,4$ and $x_3(t)$ represent the predator population. $\tau > 0$ is a constant delay due to gestation. D_i is a positive constant and denotes the dispersal rate, $i = 1,2,4$. $a_i, a_{ij}\,(i, j = 1,2,3,4)$, and m are positive constants.

We adopt the following notations and concepts throughout the rest of this work.

let $x = (x_1, x_2, x_3\, x_4) \in R_4^+ = \{x \in R^4 : x_i \geq 0, i = 1,2,3,4\}$

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The notation x > 0 denotes $x \in \text{Int}\, R^4$. For ecological reasons, we consider system (1.1), only in $\text{Int}\, R_4^+$. Let $C^+ = C([-\tau,0]; R_4^+)$ denote Banach space of all nonnegative continuous functions with

$$\|\phi\| = \sup_{s\in[-\tau,0]} |\phi(s)|, \qquad \text{for } \phi \in C^+. \qquad (1.2)$$

Then, if we choose the initial function space of system (1.1) to be C^+, it can be seen that, for any $\phi = (\phi_1,\phi_2,\phi_3,\phi_4) \in C^+$ and $\phi(0) > 0$, there exists $\alpha > 0$ and a unique $x(t,\phi)$ of system (1.1) on $[-\tau,\alpha)$, which remains positive for all $t \in [0,\alpha)$, such solutions of system (1.1) are called positive solutions. Hence, in the rest of this work , we always assume that

$$\phi \in C^+, \phi(0) > 0. \qquad (1.3)$$

Definition 1. System (1.1) is said to be uniformly persistent if there exists a compact region $D \subset \text{Int}\, R_4^+$ such that every solution $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ of system (1.1) with initial conditions (1.3), eventually enters and remains in the region *D*.

In the following, we say an equilibrium of the system is globally asymptotically stable if it attracts all positive solutions of the system.

The organization of this paper is as follows. In the next section; we present a uniform persistence results for system (1.1). In section 3, we derive the local stability. Section 4 provides sufficient conditions for the positive equilibrium of system (1.1) to be globally asymptotically stable.

2. UNIFORM PERSISTENCE

System (1.1) has a unique positive equilibrium, if and only if the following conditions are true:

$(H1)$ $a_{31} > a_3$,

$(H2)$ $ma_1 a_{31} > a_{31}(a_{31} - a_3)$.

In the following, we always assume that such a positive equilibrium exists and denote it by $E^*(x_1^*, x_2^*, x_3^*, x_4^*)$.

Lemma 2.1 Let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ denote any positive solution of system (1.1) with the initial condition (1.3). If $a_3 < a_{31}$, then there exists a $T > 0$ such that

$$x_i(t) \le M_i, \ (i = 1,2,3,4) \qquad \text{for } t \ge T, \qquad (2.1)$$

where

$$\left.\begin{aligned} &M_1 = M_2 = M_4 > M_1^*, M_3 > M_2^*, \\ &M_1^* = \max\left\{\frac{a_1}{a_{11}}, \frac{a_2}{a_{22}}, \frac{a_4}{a_{44}}\right\}, \quad M_2^* = \frac{a_{31} - a_3}{ma_3} M_1 e^{(a_{11}-a_1)\tau} \end{aligned}\right\} \quad (2.2)$$

Proof: We define $V(t) = \max\{x_1(t), x_2(t), x_4(t)\}$

Calculating the upper-right derivative of V along the positive solution of system (1.1), we have the following:

(P1) If $x_1(t) > x_2(t)$; $x_1(t) > x_4(t)$ or $x_1(t) = x_2(t) = x_4(t)$ and $\dot{x}_1(t) \ge \dot{x}_2(t)$, $\dot{x}_1(t) \ge \dot{x}_4(t)$,

$$\begin{aligned} D^+V(t) = \dot{x}_1(t) &= x_1(t)\left[a_1 - a_{11}x_1(t) - \frac{a_{13}x_3(t)}{mx_3(t) + x_1(t)}\right] \\ &\quad + D_1(x_2(t) + x_4(t) - x_1(t)) \\ &\le x_1(t)[a_1 - a_{11}x_1(t)]. \end{aligned}$$

(P2) If $x_1(t) < x_2(t)$; $x_4(t) < x_2(t)$ or $x_1(t) = x_2(t) = x_4(t)$ and $\dot{x}_1(t) \le \dot{x}_2(t)$; $\dot{x}_4(t) \le \dot{x}_2(t)$,

$$\begin{aligned} D^+V(t) = \dot{x}_2(t) &= x_2(t)[a_2 - a_{22}x_2(t)] + D_2(x_1(t) + x_4(t) - x_2(t)) \\ &\le x_2(t)[a_2 - a_{22}x_2(t)]. \end{aligned}$$

(P3) If $x_1(t) < x_4(t)$; $x_2(t) < x_4(t)$ or $x_1(t) = x_4(t) = x_2(t)$ and $\dot{x}_1(t) \le \dot{x}_4(t)$; $\dot{x}_2(t) \le \dot{x}_4(t)$,

$$\begin{aligned} D^+V(t) = \dot{x}_4(t) &= x_4(t)[a_4 - a_{44}x_4(t)] + D_4(x_1(t) + x_2(t) - x_4(t)) \\ &\le x_4(t)[a_4 - a_{44}x_4(t)]. \end{aligned}$$

From (P1)-(P3), we have

$$D^+V(t) \le x_i(t)[a_i - a_{ii}x_i(t)], i = 1 \text{ or } 2 \text{ or } 4. \qquad (2.3)$$

From (2.3), we can obtain the following:

(i) If $\max\{x_1(0), x_2(0), x_4(0)\} \le M_1$, then

$$\max\{x_1(t), x_2(t), x_4(t)\} \le M_1, \quad t \ge 0.$$

(ii) If $\max\{x_1(0), x_2(0), x_4(0)\} > M_1$, and

$$-\alpha = \max_{i=1,2,4}\{M_1(a_i - a_{ii}M_1)\}, \ (\alpha > 0),$$

we consider the following four possibilities:

(a) $V(0) = x_1(0) > M_1 \{x_1(0) > x_2(0); \quad x_1(0) > x_4(0)\}$

(b) $V(0) = x_2(0) > M_1 \{x_1(0) < x_2(0); \quad x_4(0) < x_2(0)\}$

(c) $V(0) = x_4(0) > M_1 \{x_1(0) < x_4(0); \quad x_2(0) < x_4(0)\}$

(d) $V(0) = x_1(0) = x_2(0) = x_4(0) > M_2$.

If (a) holds, then there exists $\varepsilon > 0$, such that if $t \in [0,\varepsilon)$, then $V(t) = x_1(t) > M_1$, and we have

$$D^+V(x_1(t), x_2(t), \ x_4(t)) = \dot{x}_1(t) < -\alpha < 0.$$

If (b) holds, then there exists $\varepsilon > 0$, such that if $t \in [0,\varepsilon)$, then $V(t) = x_2(t) > M_1$, and we have

$$D^+V(x_1(t), x_2(t), x_4(t)) = \dot{x}_2(t) < -\alpha < 0.$$

If (c) holds, then there exists $\varepsilon > 0$, such that if $t \in [0,\varepsilon)$, then $V(t) = x_4(t) > M_1$, and we have

$$D^+V(x_1(t), x_2(t), x_4(t)) = \dot{x}_4(t) < -\alpha < 0.$$

If (d) holds, then there exists $\varepsilon > 0,$ such that if $t \in [0, \varepsilon),$ then
$V(t) = x_1(t) > M_1$, or $V(t) = x_2(t) > M_1$
or $V(t) = x_4(t) > M_1$.
Similar to (a), (b) and (c), we have
$D^+V(x_1(t), x_2(t), x_4(t)) = \dot{x}_i(t), (i = 1 \text{ or} 2 \text{ or } 4) < -\alpha < 0.$

From what has been discussed above, we can conclude that if $V(0) > M_1$, then $V(t)$ is strictly monotone decreasing with speed at least α, therefore, there exists $T_1 > 0,$ if $t \geq T_1$, we have

$$V(t) = \max\{x_1(t), x_2(t), x_4(t)\} \leq M_1.$$

In addition, from the third equation of system (1.2) we obtain $\dot{x}_3(t) \leq (a_{31} - a_3)x_3(t)$

For $t > \tau,$ we have $x_3(t) \leq x_3(t-\tau)e^{(a_{31}-a_3)\tau}$,

which is equivalent to

$t > \tau, \quad x_3(t-\tau) \geq x_3(t)e^{(a_3-a_{31})\tau}.$

Therefore, for $t > T_1 + \tau,$ we have

$$\dot{x}_3(t) \leq x_3(t)\left[-a_3 + \frac{a_{31}M_1}{M_1 + mx_3(t-\tau)}\right]$$

$$\leq x_3(t)\left[-a_3 + \frac{a_{31}M_1}{M_1 + me^{(a_3-a_{31})\tau}x_3(t)}\right]$$

$$= x_3(t)\left[\frac{(a_{31}-a_3)M_1 - ma_3e^{(a_3-a_{31})\tau}x_3(t)}{M_1 + me^{(a_3-a_{31})\tau}x_3(t)}\right]$$

A standard comparison argument shows that $\limsup_{t\to\infty} x_3(t) \leq M_2^*$. The proof is completed.

Theorem 2.1 Suppose that system (1.1) satisfies (H1) and the following:
$(H3)\, a_1 > a_{31}/m + D_1;$
$(H4)\, a_2 > D_2;$
$(H5)\, a_4 > D_4.$

Then system (1.1) is uniformly persistent.

Proof: Suppose $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ is a solution of system (1.1), which satisfies (1.3). According to the first equation of system (1.1), if (H3) holds, then

$$\dot{x}_1(t) > x_1(t)\left[a_1 - \frac{a_{13}}{m} - D_1 - a_{11}x_1(t)\right], \qquad (2.4)$$

which implies that
$\liminf_{t\to\infty} x_1(t) \geq (a_1 - a_{13}/m - D_1)/a_{11} \equiv m_1.$

Hence, for large *t*, $x_1(t) > m_1/2,$

and $\dot{x}_3(t) \geq x_3(t)\left[-a_3 + \dfrac{a_{31}\, m/2}{mx_3(t-\tau) + m_1/2}\right].$

Using the fact that, for large t,
$x_3(t-\tau) \leq x_3(t)e^{a_3\tau},$
we have

$$\dot{x}_3(t) \geq x_3(t)\left[\frac{m_1(a_{31}-a_3)/2 - ma_3e^{a_3\tau}x_3(t)}{me^{a_3\tau}x_3(t) + m_1/2}\right], \qquad (2.5)$$

which yields
$\liminf_{t\to\infty} x_3(t) \geq m_1(a_{31} - a_3)e^{-a_3\tau}(2ma_3)^{-1} \equiv m_3.$

Therefore, for large t, we have $x_3(t) > m_3/2.$

In addition, from the second equation of system (1.1), we obtain $\dot{x}_2(t) \geq x_2(t)[a_2 - D_2 - a_{22}x_2(t)],$

which implies that $\liminf_{t\to\infty} x_2(t) \geq \dfrac{a_2 - D_2}{a_{22}} \equiv m_2.$

Thus for large *t*, we have $x_2(t) > m_2/2.$

Moreover, from the forth equation of system (1.1) we obtain

$\dot{x}_4(t) \geq x_4(t)[a_4 - D_4 - a_{44}x_4(t)],$

which implies that $\liminf_{t\to\infty} x_4(t) \geq \dfrac{a_4 - D_4}{a_{44}} \equiv m_4.$

Thus for large t, we have $x_4(t) > m_4/2.$

Now, we let

$$D = \left\{(x_1, x_2, x_3, x_4)\,\middle|\, \frac{m_i}{2} \leq x_i \leq M_i, i = 1,2,3,4\right\}. \qquad (2.6)$$

Then D is a bounded compact region in R_4^+ which has positive distance from coordinate planes.

From what has been discussed above, we obtain that there exists a $T^* > 0,$ if $t > T^*,$ then every positive solution of system (1.1) with initial condition (2.2), eventually enters and remains in the region D. The proof is completed.

3. LOCAL ASYMPTOTICAL STABILITY

Linearizing system (1.1) at $E^*(x_1^*, x_2^*, x_3^*, x_4^*)$, we obtain

$$\left.\begin{aligned} \dot{N}_1(t) &= A_{11}N_1(t) + A_{12}N_2(t) + A_{13}N_3(t) + A_{14}N_4(t),\\ \dot{N}_2(t) &= A_{21}N_1(t) + A_{22}N_2(t) + A_{24}N_4(t),\\ \dot{N}_3(t) &= B_{31}N_1(t-\tau) + B_{33}N_3(t-\tau),\\ \dot{N}_4(t) &= A_{41}N_1(t) + A_{42}N_2(t) + A_{44}N_4(t) \end{aligned}\right\} \qquad (3.1)$$

where

$$A_{11} = -\frac{D_1x_2^*}{x_1^*} - \frac{D_1x_4^*}{x_1^*} - a_{11}x_1^* + \frac{a_{13}a_3(a_{31}-a_3)}{ma_{31}}, A_{12} = D_1, A_{13} = -a_{13}\left(\frac{a_3}{a_{31}}\right)^2, A_{14} = D_1,$$

$$A_{21} = D_2, A_{22} = -\frac{D_2x_1^*}{x_2^*} - \frac{D_2x_4^*}{x_2^*} - a_{22}x_2^*, A_{24} = D_2.$$

$$B_{31} = \frac{(a_{31}-a_3)^2}{ma_{31}}, B_{33} - \frac{a_3(a_{31}-a_3)}{a_{31}}.$$

$$A_{14} = D_4, A_{24} = D_4, A_{44} = -\frac{D_4x_1^*}{x_4^*} - \frac{D_4x_2^*}{x_4^*} - a_{44}x_4^*.$$

Theorem 3.1. Suppose that system (1.1) satisfies $(H1),(H2)$ and the following:

$(H6)\ -(2A_{11}+A_{12}+A_{14}+A_{21}+A_{41})/A_{13}+2\tau(B_{31}-B_{33})<0,$
$(H7)\ 2A_{11}+A_{12}+A_{21}+A_{24}+A_{42}<0,$
$(H8)\ 2A_{44}+A_{14}+A_{24}+A_{41}+A_{42}<0,$
$(H9)\ \tau(B_{31}-B_{33})<1.$

Then the positive equilibrium E^* of (1.1) is locally asymptotically stable.

Proof: The third equation of (3.1) can be rewritten as

$$\frac{d}{dt}\left[N_3(t)+B_{31}\int_{t-\tau}^{t}N_1(s)ds+B_{33}\int_{t-\tau}^{t}N_3(s)ds\right] = B_{31}N_1(t)+B_{33}N_3(t). \quad (3.2)$$

Define

$$W_{31}(N)(t)=\left[N_3(t)+B_{31}\int_{t-\tau}^{t}N_1(s)ds+B_{33}\int_{t-\tau}^{t}N_3(s)ds\right]^2. \quad (3.3)$$

Then, along the solution of (3.1), we have

$$\frac{d}{dt}W_{31}(N)(t)=2[B_{31}N_1(t)+B_{33}N_3(t)]\left[N_3(t)+B_{31}\int_{t-\tau}^{t}N_1(s)ds+B_{33}\int_{t-\tau}^{t}N_3(s)ds\right]$$
$$=2B_{31}N_1(t)N_3(t)+2B_{33}N_3^2(t)+2B_{31}^2N_1(t)\int_{t-\tau}^{t}N_1(s)ds$$
$$+2B_{31}B_{33}N_3(t)\int_{t-\tau}^{t}N_1(s)ds+2B_{31}B_{33}N_1(t)\int_{t-\tau}^{t}N_3(s)ds$$
$$+2B_{33}^2N_3(t)\int_{t-\tau}^{t}N_3(s)ds.$$

Using the Cauchy-Schwarz inequality and the inequality
$a^2+b^2\geq 2ab$, we get

$$\frac{d}{dt}W_{31}(N)(t)\leq 2B_{31}N_1(t)N_3(t)+2B_{33}N_3^2(t)+\tau(B_{31}-B_{33})(B_{31}N_1^2(t)-B_{33}N_3^2(t))$$
$$+(B_{31}-B_{33})\left[B_{31}\int_{t-\tau}^{t}N_1^2(s)ds-B_{33}\int_{t-\tau}^{t}N_3^2(t)ds\right]. \quad (4.4)$$

Now let $W_3(N)(t)$ be defined by

$$W_3(t)=W_3(N)(t)=W_{31}(N)(t)+W_{32}(N)(t), \quad (4.5)$$

$$W_{32}(N)(t)=(B_{31}-B_{33})\left[B_{31}\int_{t-\tau}^{t}\int_{v}^{t}N_1^2(s)dsdv-B_{33}\int_{t-\tau}^{t}\int_{v}^{t}N_3^2(t)dsdv\right]. \quad (3.6)$$

Then we derive from (3.4)-(3.6) that

$$\frac{d}{dt}W_3(N)(t)\leq 2B_{31}N_1(t)N_3(t)+2B_{33}N_3^2(t)+2\tau(B_{31}-B_{33})(B_{31}N_1^2(t)-B_{33}N_3^2(t))$$

Let

$$W(t)=W(N)(t)=-\frac{B_{31}}{A_{13}}[N_1^2(t)+N_2^2(t)+N_4^2(t)]+W_3(t),$$

then, along the solution of (3.1), we have

$$\left.\begin{aligned}
\frac{d}{dt}W(t)\leq\ & -2\frac{B_{31}}{A_{13}}N_1(t)\begin{bmatrix}A_{11}N_1(t)+A_{12}N_2(t)\\ +A_{13}N_3(t)+A_{14}N_4(t)\end{bmatrix}\\
& -2\frac{B_{31}}{A_{13}}N_2(t)[A_{21}N_1(t)+A_{22}N_2(t)+A_{24}N_4(t)]\\
& -2\frac{B_{31}}{A_{13}}N_4(t)[A_{41}N_1(t)+A_{42}N_2(t)+A_{44}N_4(t)]\\
& +2B_{31}N_1(t)N_3(t)+2B_{33}N_3^2(t)\\
& +2\tau(B_{31}-B_{33})(B_{31}N_1^2(t)-B_{33}N_3^2(t))\\
\leq & -2\frac{B_{31}}{A_{13}}\begin{bmatrix}A_{11}N_1^2(t)+A_{22}N_2^2(t)+A_{44}N_4^2(t)\\ +(A_{12}+A_{21})(N_1^2+N_2^2)\end{bmatrix}\\
& -2\frac{B_{31}}{A_{13}}\begin{bmatrix}(A_{14}+A_{41})(N_1^2(t)+N_4^2(t))\\ +(A_{24}+A_{42})(N_1^2(t)+N_4^2(t))\end{bmatrix}\\
& +2B_{33}N_3^2(t)+2\tau(B_{31}-B_{33})\\
& (B_{31}N_1^2(t)-B_{33}N_3^2(t))
\end{aligned}\right\} \quad (3.7)$$

Using the inequality $a^2+b^2\geq 2ab$, we have

$$\frac{d}{dt}W(t)\leq-\frac{B_{31}}{A_{13}}\begin{bmatrix}2A_{11}+A_{12}+A_{14}+A_{21}+A_{41}\\ +2\tau(B_{31}-B_{33})B_{31}\end{bmatrix}N_1^2$$
$$-\frac{B_{31}}{A_{13}}[2A_{22}+A_{12}+A_{21}+A_{24}+A_{42}]N_2^2$$
$$+[2B_{33}-2\tau(B_{31}-B_{33})B_{33}]N_3^2$$
$$-\frac{B_{31}}{A_{13}}[2A_{44}+A_{14}+A_{41}+A_{24}+A_{42}]N_4^2.$$

Then, we have

$$\frac{d}{dt}W(t)\leq-\alpha_1N_1^2(t)-\alpha_2N_2^2(t)-\alpha_3N_3^2(t)-\alpha_4N_4^2(t), \quad (3.8)$$

in which

$$\alpha_1=\frac{B_{31}(2A_{11}+A_{12}+A_{14}+A_{21}+A_{41})}{A_{13}}-2\tau(B_{31}-B_{33})B_{31},$$
$$\alpha_2=\frac{B_{31}}{A_{13}}(2A_{22}+A_{12}+A_{21}+A_{24}+A_{42})$$
$$\alpha_3=-2B_{33}+2\tau(B_{31}-B_{33})B_{33},$$
$$\alpha_4=\frac{B_{31}}{A_{14}}(2A_{44}+A_{14}+A_{24}+A_{41}+A_{42})$$

Clearly, assumptions (H6)-(H9) imply that $\alpha_1>0,\alpha_2>0,\alpha_3>0,\alpha_4>0.$

Denote $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then (3.8) leads to

$$W(t) + \alpha \int_T^t \left[N_1^2(s) + N_2^2(s) + N_3^2(s) + N_4^2(s)\right] ds \le W(t),$$

for $t \ge T$, (3.9)

and which implies

$N_1^2(t) + N_2^2(t) + N_3^2(t) + N_4^2(t) \in L^1(T, \infty)$.

It is easy to see from (3.1) and the boundedness of $N(t)$ that $\sum_{i=1}^{4} N_i^2(t)$ is uniformly continuous and then, using Barabalates **Lemma** (Gopalsamy, 1992), we can conclude that $\lim_{t\to\infty} \sum_{i=1}^{4} N_i^2 = 0.$ Therefore, the zero solution of (3.1) is asymptotically stable and this completes the proof.

Remark: We remark here that, from the proof of Theorem (3.1), it is easy to know that, under the Assumptions (H1) and (H2), if

$2A_{11} + A_{12} + A_{21} + A_{14} + A_{41} < 0,$

$2A_{22} + A_{12} + A_{21} + A_{24} + A_{42} < 0,$ and

$2A_{44} + A_{14} + A_{41} + A_{24} + A_{42} < 0,$

then the positive equilibrium of the "instantaneous" (when $\tau = 0$) model (1.1) is locally asymptotically stable. If

$2A_{22} + A_{12} + A_{21} + A_{24} + A_{42} < 0,$ and

$2A_{44} + A_{14} + A_{41} + A_{24} + A_{42} < 0,$

then the local stability of E^* of (1.1) is preserved for small τ satisfying (H6) and (H9).

4. GLOBAL ASYMPTOTIC STABILITY

In this section, we proceed to the study of global attractively of positive equilibrium of system (1.1). To achieve this, we need the following theorem. But let us first consider an autonomous system of delay differential equation defined as

$$\dot{x}(t) = F(x_t), \qquad (4.1)$$

such that $F(0) = 0$ and $F : C([-\tau, 0], R^n) \to R^n, \tau > 0$, is Lipschitzian, where $C = C([-\tau, 0], R^n)$ is the set of continuous functions defined on $[-\tau, 0]$, with the norm $\|\phi\| = \max_{-\tau\le\theta\le 0} |\phi(\theta)|$, and where $|\bullet|$ is any norm in R^n.

Theorem A (Kuang, 1993) Let $w_1(\bullet)$, $w_2(\bullet)$ and $w_4(\bullet)$ be nonnegative continuous scalar functions such that $w_i(0) = 0, i = 1,2,4; w_2(r) > 0,\ w_4(r) > 0$ for $r > 0, \lim_{r\to+\infty} w_1(r) = +\infty$, and $V: C \to R$ is a continuously differentiable scalar functional for a special set *S* of solutions of (1.1), and the following are satisfied

(1) $V(\phi) \ge w_1(\phi(0))$,

(2) $\dot{V}(\phi)\big|_{(5.1)} \le -w_i(|\phi(0)|),\ i = 2,4.$

Then *x* = 0 is asymptotically stable with respect to the set *S*. That is, solutions that stay in *S* converge to $x = 0$.

Our strategy in the proof of global asymptotic stability of the positive equilibrium of (1.1) is to construct a suitable Lyapunov functional. Let $P(u)$ be defined by

$$p(u) = \frac{u}{m+u},$$

then system (1.1) can be rewritten as

$$\left.\begin{aligned}
\dot{x}_1 &= x_1\left\{-a_{11}(x_1 - x_1^*) + a_{13}\left[\begin{matrix}\dfrac{x_3^*}{x_1^*} P\left(\dfrac{x_1^*}{x_3^*}\right) - \\ \dfrac{x_3}{x_1} P\left(\dfrac{x_1}{x_3}\right)\end{matrix}\right]\right\} \\
&\quad - \frac{D_1}{x_1^*} x_2 x_4 (x_1 - x_1^*) + \frac{D_1}{x_1^*} x_1 x_4 (x_2 - x_2^*) \\
&\quad + \frac{D_1}{x_1^*} x_1 x_2 (x_4 - x_4^*), \\
\dot{x}_2 &= x_2\{-a_{22}(x_2 - x_2^*)\} - \frac{D_2}{x_2^*} x_1 x_4 (x_2 - x_2^*) \\
&\quad + \frac{D_2}{x_2^*} x_2 x_4 (x_1 - x_1^*) \\
&\quad + \frac{D_2}{x_2^*} x_1 x_2 (x_4 - x_4^*), \\
\dot{x}_3 &= a_{31} x_3 \left[P\left(\frac{x_1(t-\tau)}{x_3(t-\tau)}\right) - P\left(\frac{x_1^*}{x_3^*}\right)\right], \\
\dot{x}_4 &= x_4\{-a_{44}(x_4 - x_4^*)\} - \frac{D_4}{x_4^*} x_1 x_2 (x_4 - x_4^*) \\
&\quad + \frac{D_4}{x_4^*} x_2 x_4 (x_1 - x_1^*) \\
&\quad + \frac{D_4}{x_4^*} x_1 x_2 (x_4 - x_4^*).
\end{aligned}\right\} \qquad (4.2)$$

Define ,

$$u = \frac{x_1}{x_3}, \quad u^* = \frac{x_1^*}{x_3^*} \ ,$$

then system (4.2) becomes

$$\left.\begin{aligned}
\dot{x}_1 &= x_1\left\{-a_{11}\left(x_1-x_1^*\right)+a_{13}\left[\frac{P(u^*)}{u^*}-\frac{P(u)}{u}\right]\right\}\\
&\quad -\frac{D_1}{x_1^*}x_2x_4\left(x_1-x_1^*\right)\\
&\quad +\frac{D_1}{x_1^*}x_1x_4\left(x_2-x_2^*\right)+\frac{D_1}{x_1^*}x_1x_2\left(x_4-x_4^*\right),\\
\dot{u} &= u\{-a_{11}\left(x_1-x_1^*\right)+a_{31}\left[\frac{P(u^*)}{u^*}-\frac{P(u)}{u}\right]\\
&\quad -a_{13}\left[P(u(t-\tau))-P(u^*)\right]\\
&\quad +\frac{D_1}{x_1^*x_1}x_2x_4\left(x_1-x_1^*\right)+\frac{D_1}{x_1^*}x_4\left(x_2-x_2^*\right)\\
&\quad +\frac{D_1}{x_1^*}x_2\left(x_4-x_4^*\right),\\
\dot{x}_4 &= x_4\left\{-a_{44}\left(x_4-x_4^*\right)\right\}-\frac{D_4}{x_4^*}x_1x_2\left(x_4-x_4^*\right)\\
&\quad +\frac{D_4}{x_4^*}x_2x_4\left(x_1-x_1^*\right)\\
&\quad +\frac{D_4}{x_4^*}x_1x_2\left(x_4-x_4^*\right).
\end{aligned}\right\} \tag{4.3}$$

Define $v(t)=(v_1(t),v_2(t),v_3(t),v_4(t))$ where $v_1(t)=x_1(t)-x_1^*, v_2(t)=x_2(t)-x_2^*, v_3(t)=u(t)-u^*, v_4(t)=x_4(t)-x_4^*$

$$F(v_3)=p(u)-P(u^*)=\frac{mv_3}{(m+u^*)(m+u)}\ . \tag{4.4}$$

Observing that

$v_3F(v_3)>0, v_3\neq 0;$

$$F'(v_3)=\frac{mu}{(m+u)^2}<1, \tag{4.5}$$

It is easy to prove that

$$\left[\frac{P(u^*)}{u^*}-\frac{P(u)}{u}\right]=\frac{1}{m}F(v_3).$$

Therefore, from (4.3) and (4.4), we finally obtain

$$\left.\begin{aligned}
\dot{v}_1 &= (v_1+x_1^*)\left[-a_{11}v_1+\frac{a_{13}}{m}F(v_3)\right]-\frac{D_1}{x_1^*}x_2x_4v_1\\
&\quad +\frac{D_1}{x_1^*}x_1x_4v_2+\frac{D_1}{x_1^*}x_1x_2v_4,\\
\dot{v}_2 &= -a_{22}v_2\left(v_2+x_2^*\right)-\frac{D_2}{x_2^*}x_1x_4v_2+\frac{D_2}{x_2^*}x_2x_4v_1,\\
\dot{v}_3 &= (v_3+u^*)[-a_{11}v_1+\frac{a_{13}}{m}F(v_3)\\
&\quad -a_{31}F(v_3(t-\tau))+\frac{D_1}{x_1^*}x_4v_2\\
&\quad -\frac{D_1}{x_1^*x_1}x_2x_4v_1+\frac{D_1}{x_1^*}x_2v_4],\\
\dot{v}_4 &= -a_{44}v_4\left(v_4+x_4^*\right)-\frac{D_4}{x_4^*}x_1x_2v_4+\frac{D_4}{x_4^*}x_2x_4v_1.
\end{aligned}\right\} \tag{4.6}$$

Now we formulate the result on the global stability of the equilibrium E^* of (1.1) as follows.

Theorem 4.1. Suppose that system (1.1) satisfies (H1)-(H5) and the following (H10) $A_1>0,1,2,3,4,$ where

$$A_1=\frac{ma_{11}^2}{a_{13}}-\frac{D_1x_4^*M_2}{x_1^*M_1}-\frac{1}{2}\tau a_{31}\left(a_{11}+\frac{2D_1x_4^*M_2}{x_1^*m_1}\right),$$

$$A_2=\frac{ma_{11}D_1x_2^*a_{22}}{a_{13}D_2x_1^*}-\frac{D_1M_4}{2x_1^*}-\frac{\tau a_{31}D_1M_4}{2x_1^*},$$

$$A_3=a_{31}-\frac{a_{13}}{m}-\frac{D_1M_4}{2x_1^*}-\frac{D_1M_2x_4^*}{x_1^*m_1}-\frac{\tau D_1M_2}{2x_1^*}$$
$$-\frac{1}{2}\tau a_{31}\left(a_{11}+\frac{2a_{13}}{m}+2a_{31}+\frac{D_1M_4}{x_1^*}+\frac{2D_1M_2x_2^*}{mx_1^*}+\frac{D_1M_2}{x_1^*}\right),$$

$$A_4=\frac{ma_{11}D_1x_4^*a_{44}}{a_{13}D_4x_1^*}-\frac{D_1M_2}{2x_1^*}-\frac{\tau a_{31}D_1M_2}{2x_1^*},$$

then the positive equilibrium $E^*(x_1^*,x_2^*,x_3^*,x_4^*)$ of (1.1) is globally asymptotic stable.

Proof: To prove that the global asymptotic stability of the positive equilibrium of E^* of (1.1) is equivalent to that of the trivial solution of (4.6), let

$$V_1(t)=\sum_{i=1,2,4}c_i\left(v_i-x_i\ln\frac{v_i+x_i^*}{x_i^*}\right)+\int_{u^*}^{u}\frac{P(v)-P(u^*)}{v}\,dv, \tag{4.7}$$

where,

$$c_1=\frac{ma_{11}}{a_{13}}, c_2=\frac{mD_1a_{11}x_2^*}{D_2a_{13}x_1^*}, c_4=\frac{mD_1a_{11}x_4^*}{D_4a_{13}x_1^*}.$$

Along the solution of (4.6), we have

$$\begin{aligned}
\frac{d}{dt}V_1(t) &= \sum_{i=1,2,4}c_i\frac{v_i}{v_i+x_i^*}\dot{v}_i(t)+\frac{F(v_3)}{u}\dot{v}_3(t)\\
&= -c_1a_{11}v_1^2(t)-\frac{c_1D_1}{x_1^*}\left(\sqrt{\frac{x_2x_4}{x_1}}v_1-\sqrt{\frac{x_1x_4}{x_2}}v_2\right)^2-c_2a_{22}v_2^2(t)\\
&\quad -\frac{D_2c_2v_2^2x_1x_4}{x_2^*x_2}+\frac{D_2c_2}{x_2^*}v_1v_4x_4-c_4a_{44}v_4^2-\frac{D_4c_4}{x_4^*x_4}v_4^2x_1x_2\\
&\quad +\frac{D_4c_4}{x_4^*}v_1v_4x_2+\frac{a_{13}}{m}F^2(v_3(t))-a_{31}F(v_3)F(v_3(t-\tau))\\
&\quad +\frac{D_1}{x_1^*}x_4v_2F(v_3)+\frac{D_1}{x_1^*}x_2v_4F(v_3)-\frac{D_1x_2x_4^*}{x_1^*x_1}v_1F(v_3)\\
&= -c_1a_{11}v_1^2(t)-c_2a_{22}v_2^2(t)-c_4a_{44}v_4^2(t)\\
&\quad -\frac{c_1D_1}{x_1^*}\left(\sqrt{\frac{x_2x_4}{x_1}}v_1-\sqrt{\frac{x_1x_4}{x_2}}v_2\right)^2+\frac{2D_1c_1x_2v_4}{x_1^*}\left(2v_1-x_1v_4\right)\\
&\quad +\left(\frac{a_{13}}{m}-a_{31}\right)F^2(v_3(t))+\frac{D_1}{x_1^*}x_4v_2F(v_3(t))-\frac{D_1x_2x_4^*}{x_1^*x_1}v_1F(v_3(t))\\
&\quad +\frac{D_1}{x_1^*}x_2v_4F(v_3(t))+a_{31}F(v_3(t))\int_{t-\tau}^{t}F'(v_3(s))v'(s)\,ds
\end{aligned}$$

$$\left.\begin{aligned}
&= -c_1a_{11}v_1^2(t) - c_2a_{22}v_2^2(t) - c_4a_{44}v_4^2(t)\\
&\quad -\frac{c_1D_1}{x_1^*}\left(\sqrt{\frac{x_2x_4}{x_1}}v_1 - \sqrt{\frac{x_1x_4}{x_2}}v_2\right)^2\\
&\quad +\frac{2D_1c_1x_2v_4}{x_1^*}(2v_1 - x_1v_4)\\
&\quad +\left(\frac{a_{13}}{m} - a_{31}\right)F^2(v_3(t)) + \frac{D_1}{x_1^*}x_4v_2F(v_3(t))\\
&\quad -\frac{D_1x_2x_4^*}{x_1^*x_1}v_1F(v_3(t))\\
&\quad +\frac{D_1}{x_1^*}x_2v_4F(v_3(t)) + a_{31}F(v_3(t))\int_{t-\tau}^{t}F'(v_3(s))u(s)[-a_{11}v_1(s)\\
&\quad +\frac{a_{13}}{m}F(v_3(s)) - a_{31}F(v_3(s-\tau)) + \frac{D_1}{x_1^*}x_4v_2(s) - \frac{D_1x_2x_4^*}{x_1^*x_1}v_1(s)\\
&\quad +\frac{D_1}{x_1^*}x_2v_4(s)]\,ds.
\end{aligned}\right\}\quad (4.8)$$

Using the inequality $a^2+b^2\geq 2ab$, and the Cauchy-Schwarz inequality, then from (2.6) and (4.5) we derive, for $t>T^*$, that

$$\begin{aligned}
\frac{d}{dt}V_1(t) &= -c_1a_{11}v_1^2(t) - c_2a_{22}v_2^2(t) - c_4a_{44}v_4^2(t)\\
&\quad -\frac{c_1D_1}{x_1^*}\left(\sqrt{\frac{x_2x_4}{x_1}}v_1 - \sqrt{\frac{x_1x_4}{x_2}}v_2\right)^2 + \frac{2D_1c_1x_2v_4}{x_1^*}(2v_1 - x_1v_4)\\
&\quad +\left(\frac{a_{13}}{m} - a_{31}\right)F^2(v_3(t)) + \frac{D_1}{x_1^*}x_4v_2F(v_3(t)) - \frac{D_1x_2x_4^*}{x_1^*x_1}v_1F(v_3(t))\\
&\quad +\frac{D_1}{x_1^*}x_2v_4F(v_3(t)) + a_{31}F(v_3(t))\int_{t-\tau}^{t}F'(v_3(s))u(s)[-a_{11}v_1(s)\\
&\quad +\frac{a_{13}}{m}F(v_3(s)) - a_{31}F(v_3(s-\tau)) + \frac{D_1}{x_1^*}x_4v_2(s) - \frac{D_1x_2x_4^*}{x_1^*x_1}v_1(s)\\
&\quad +\frac{D_1}{x_1^*}x_2v_4(s)]\,ds
\end{aligned}$$

$$\left.\begin{aligned}
\frac{d}{dt}V(t) &\leq -c_1a_{11}v_1^2(t) - c_2a_{22}v_2^2(t) - c_4a_{44}v_4^2(t)\\
&\quad -\frac{c_1D_1}{x_1^*}\left(\sqrt{\frac{x_2x_4}{x_1}}v_1 - \sqrt{\frac{x_1x_4}{x_2}}v_2\right)^2 + \frac{2D_1c_1x_2v_4}{x_1^*}(2v_1 - x_1v_4)\\
&\quad +\left(\frac{a_{13}}{m} - a_{31}\right)F^2(v_3(t)) + \frac{D_1M_4}{2x_1^*}\left(v_2^2(t) + F^2(v_3(t))\right)\\
&\quad +\frac{D_1M_2x_4^*}{2x_1^*m_1/2}\left(v_1^2(t) + F^2(v_3(t))\right) + \frac{D_1M_2}{2x_1^*}\left(v_4^2 + F^2(v_3(t))\right)\\
&\quad +a_{31}|F(v_3(t))|\int_{t-\tau}^{t}\frac{F'(v_3(s))u(s)}{[a_{11}|v_1(s)|} + \frac{a_{13}}{m}|F(v_3(s))|\\
&\quad -a_{31}|F(v_3(t-\tau))| + \frac{D_1}{x_1^*}M_4|v_2(s)| + \frac{D_1M_2x_4^*}{x_1^*m/2}|v_1(s)|\\
&\quad +\frac{D_1M_2}{x_1^*}|v_4(s)|]\,ds
\end{aligned}\right\}\quad (4.9)$$

$$\left.\begin{aligned}
&\leq \left(-c_1a_{11} + \frac{D_1M_2x_4^*}{x_1^*m_1}\right)v_1^2(t) + \left(-c_2a_{22} + \frac{D_1M_4}{2x_1^*}\right)v_2^2(t)\\
&\quad +\left(-c_4a_{44} + \frac{D_1M_2}{2x_1^*}\right)v_4^2(t) - \frac{c_1D_1}{x_1^*}\left(\sqrt{\frac{x_2x_4}{x_1}}v_1 - \sqrt{\frac{x_1x_4}{x_2}}v_2\right)^2\\
&\quad +\frac{2c_1D_1M_2}{x_1^*}|v_4(t)|\left(2|v_1(t)| - x_1|v_4(t)|\right) + \left[\frac{a_{13}}{m} - a_{31} + \frac{D_1M_4}{2x_1^*}\right.\\
&\quad \left.+\frac{D_1M_2x_4^*}{x_1^*m_1} + \frac{D_1M_2}{2x_1^*}\right]F^2(v_3(t)) + a_{31}|F(v_3(t))|\int_{t-\tau}^{t}[a_{11}|v_1(s)|\\
&\quad +\frac{a_{13}}{m}|F(v_3(s))| + a_{31}|F(v_3(t-\tau))| + \frac{D_1M_4}{x_1^*}|v_2(s)|\\
&\quad +\frac{2D_1M_2x_4^*}{x_1^*m_1}|v_1(s)| + \frac{D_1M_2}{x_1^*}|v_4(s)|]\,ds\\
&\leq \left(-c_1a_{11} + \frac{D_1M_2x_4^*}{x_1^*m_1}\right)v_1^2(t) + \left(-c_2a_{22} + \frac{D_1M_4}{2x_1^*}\right)v_2^2(t)\\
&\quad +\left(-c_4a_{44} + \frac{D_1M_2}{2x_1^*}\right)v_4^2(t) - \frac{c_1D_1}{x_1^*}\left(\sqrt{\frac{x_2x_4}{x_1}}v_1 - \sqrt{\frac{x_1x_4}{x_2}}v_2\right)^2\\
&\quad +\frac{2c_1D_1M_2}{x_1^*}|v_4(t)|\left(2|v_1(t)| - x_1|v_4(t)|\right) + \left[\frac{a_{13}}{m} - a_{31} + \frac{D_1M_4}{2x_1^*}\right.\\
&\quad \left.+\frac{D_1M_2x_4^*}{x_1^*m_1} + \frac{D_1M_2}{2x_1^*}\right]F^2(v_3(t)) + \frac{1}{2}a_{31}\tau\int_{t-\tau}^{t}[a_{11}v_1^2(s)\\
&\quad +\frac{a_{13}}{m}F^2(v_3(s)) + a_{31}F^2(v_3(t-\tau)) + \frac{D_1M_4}{x_1^*}v_2^2(s)\\
&\quad +\frac{2D_1M_2x_4^*}{x_1^*m_1}v_1^2(s) + \frac{D_1M_2}{x_1^*}v_4^2(s)]\,ds.
\end{aligned}\right\}$$

Now define Lyapounov functional $V(t)$ as

$$\left.\begin{aligned}
V(t) &= V_1(t) + \frac{1}{2}a_{31}\int_{t-\tau}^{t}\int_{v}^{t}\left[a_{11}v_1^2(s) + \frac{a_{13}}{m}F^2(v_3(s)) + a_{31}F^2(v_3(s-\tau))\right.\\
&\quad \left.+\frac{D_1M_4}{x_1^*}v_2^2(s) + \frac{2D_1M_2x_4^*}{x_1^*m_1}v_1^2 + \frac{D_1M_2}{x_1^*}v_4^2(s)\right]\\
&\quad ds\,dv + \frac{1}{2}\tau a_{31}^2\int_{t-\tau}^{t}F^2(v_3(s))\,ds.
\end{aligned}\right\}\quad (4.10)$$

Then we have from (4.7), (4.9) and (4.10) that for $t\geq T^*$,

$$\left.\begin{aligned}
\frac{d}{dt}V(t) &\leq -A_1v_1^2(t) - A_2v_2^2(t) - A_3F^2(v_3(t)) - A_4v_4^2(t)\\
&\quad -\frac{c_1D_1}{x_1^*}\left(\sqrt{\frac{x_2x_4}{x_1}}v_1 - \sqrt{\frac{x_1x_4}{x_2}}v_2\right)^2\\
&\quad +\frac{2c_1D_1M_2}{x_1^*}|v_4|\left(2|v_1| - x_1|v_4|\right)
\end{aligned}\right\}\quad (4.11)$$

$$\leq -A_1v_1^2(t) - A_2v_2^2(t) - A_3F^2(v_3(t)) - A_4v_4^2(t).$$

Define $w_1(\|v(t)\|) = V_1(t)$

where $w_1(\bullet)$ is a continuous positive definite function of $s, s \geq 0$, such that

$w_1(0) = 0$ and $w_1(s) \to +\infty$ as $s \to +\infty$. Then, hypothesis (1) of Theorem A [5]

holds for any $(x_1, x_2, u, x_4) \in R_4^+$.

Furthermore, we see from (4.11) that $V'(t)|_{(5.6)}$ is negative definite for any $(x_1, x_2, u, x_4) \in R_4^+$ provided that $A_i > 0 (i = 1,2,3,4)$.

Therefore,

$$V'(t)|_{(5.6)} \leq -w_2(\|v(t)\|), \tag{4.12}$$

where $w_2(\bullet)$ is positive definite of $s, s \geq 0$ such that $\lim_{s \to +\infty} w_2(s) = +\infty$. And

$V'(t)|_{(5.6)} \leq -w_4(\|v(t)\|)$,

where $w_4(\bullet)$ is positive definite of $s, s \geq 0$ such that $\lim_{s \to +\infty} w_2(s) = +\infty$. Hence, hypothesis (2) of **Theorem A (Kuang, 1993)** holds, which implies the global asymptotic stability of the equilibrium E^* of (1.1) with respect to positive solutions. The proof is complete.

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الاستقرار والمثابرة لثلاثة أصناف في نظام ذي تأخير زمني

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ملخـص

يهدف البحث إلى إيضاح بعض الخواص الكيفية لحلول المعادلات التفاضلية الدالية. وتركز الدراسة على بيان خاصية الاستقرار والمثابرة لثلاثة أصناف في نظام ذي تأخير زمني. ونثبت أن للنظام صفة الاستمرار تحت شروط محددة.

الكلمـات الدالـة: نظام ذو تأخير زمني، تأخير زمني، الانتشار، المثابرة الموحدّة.

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